# 学位申請論文

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主論文

# Cancellation of Lattices and Approximation Properties of Division Algebras.

By Aiichi YAMASAKI

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## 0. Introduction

Let $R$ be a Dedekind domain with the quotient field $K$. Let $\Lambda$ be an $R$-order. In this general setting, it is proved in [3] that Roiter-Jacobinski type Divisibility Theorem holds for $\Lambda$-lattices. As a consequence, for a $\Lambda$-lattice $L$, the following two cancellation properties are equivalent.

(c) If $L'$ is a local direct summand of $nL = L \oplus \cdots \oplus L$ for some $n \geq 0$, then $L \oplus L' \simeq M \oplus L'$ implies $L \simeq M$.

(c$'$) If $L \oplus nL \simeq M \oplus nL$ for some $n \geq 0$, then $L \simeq M$.

As was pointed out in [3], putting $\Gamma := \mathrm{End}_\Lambda L$ and $B := K\Gamma$, there is an intimate connection between cancellation property and the approximation property of the group of Vaserstein $\widetilde{E}(\widehat{B})$ in the idele topology of $\widehat{B}^\times$, of which precise definitions will be recalled in §1.

Here we only indicate, $\widehat{R} := \prod R_p$, the direct product of $p$-adic completions over all maximal ideals of $R$, $\widehat{M} := M \otimes_R \widehat{R}$ for any $R$-alegbra $M$, and $\widetilde{E}(C) :=< (1+xy)(1+yx)^{-1} \mid x, y \in C,\ 1+xy \in C^\times >$ for any ring $C \ni 1$. Our first remark is

**Proposition 1** (proof in 1.5) *The property* (c′) *for* $L$ *is equivalent with the following property* (c″) *of* $\Gamma$

(c″) $\widetilde{E}(\widehat{B}) \subset \widehat{\Gamma}^\times B^\times$ *as subsets of* $\widehat{B}^\times$

## 0.1

We shall consider, for any finite dimensional $K$-algebra $B$, the following three *approximation properties over* $R$, in the idele topology of $\widehat{B}^\times$

(a) Strong approximation property:

$\widetilde{E}(B)$ is dense in $\widetilde{E}(\widehat{B})$

(a′) $B^\times$-approximation property:

$\widetilde{E}(\widehat{B})$ is contained in the closure of $B^\times$

(a″) $\widehat{R}^\times B^\times$-approximation property:

$\widetilde{E}(\widehat{B})$ is contained in the closure of $\widehat{R}^\times B^\times$

There are the obvious implications (a) $\Rightarrow$ (a′) $\Rightarrow$ (a″). Our second (rather obvious) remark is

**Proposition 2** (proof in 1.2) *The property* (a″) *for* $B$ *is equivalent with the (validity of) property* (c″) *for any* $\Lambda$*-lattice* $L$ *such that* $K\mathrm{End}_\Lambda L \simeq B$.

In the following cases, the property (a) always holds.

(1) $B$ is commutative (since $\widetilde{E}(B) = \widetilde{E}(\widehat{B}) = 1$, by definitions).

(2) $R$ is semi-local (by the Chinese Remainder Theorem).

(3) $B = M_n(C)$ by some $K$-algebra $C$ $(n \geq 2)$ (cf [3]).

## 0.2

We shall give the following reduction to division algebras.

**Theorem 1** (proof in 2.3) *Writing as* $B/J(B) = \bigoplus_{i=1}^{m} M_{n_i}(D_i)$, *with the Jacobson radical* $J(B)$ *and the division algebras* $D_i$, *in such an ordering that* $n_i = 1\,(1 \leq i \leq r)$ *and* $n_i \geq 2\,(r < i \leq m)$, *we have* *(i)* (a$'$) *for* $B \Leftrightarrow$ (a$'$) *for* $D_i\,(1 \leq i \leq r)$.

*(ii)* (a) *(resp.* (a$''$)*) for* $B \Rightarrow$ (a) *(resp.* (a$''$)*) for* $D_i\,(1 \leq i \leq r)$.

Thus the approximation properties of general $B$ can be reduced, more or less, to that of non-commutative division algebras over non-semi-local $R$, and then under a reasonable restriction, to that of central division ones, by 1.6.

Since PF-fields are the most familiar and important source of non semi-local Dedekind domains, now we restrict our attention to central division algebras over PF-fields and recall some basic facts and known results.

## 0.3

Assume that $K$ is a PF-field in the sense of Artin [1], Chap.12, and let $D$ be a finite dimensional non-commutative central division algebra over $K$.

In particular, there is given a set of valuations $\mathfrak{V}$ of $K$, satisfying the product formula $\prod_{\mathfrak{V}} |x|_v = 1$ for any $x \in K^\times$ In fact $K$ is either a number field or a function field (of one variable) over the constant field $K_0 := \{x \in K \,|\, |x|_v \leq 1 \text{ for any } v \in \mathfrak{V}\}$. $K$ is called a global field if either it is a number field or a function field with $\#K_0 < \infty$.

(i) Let $P$ be a proper non-empty subset of $\mathfrak{V}$ consisting of non-archimedean valuations. Then $R(P) := \{x \in K \,|\, |x|_p \leq 1 \text{ for any } p \in P\}$ is a Dedekind domain (with an additional requirement $R(P) \supset K_0$, if $K$ is a function field) having $K$ as its quotient field. Conversely, any such Dedekind domain $R$ in $K$ is obtained as $R = R(P)$ by some $P$

Consider the following condition (EC) for $D$ over $R = R(P)$, which is known as Eichler's condition when $K$ is a global field.

(EC) There is at least one $v \in \mathfrak{V} \setminus P$, such that the completion $D_v = D \otimes_K K_v$ is not a division algebra.

(ii) If $K$ is a global field, by Wang-Platonov Theorem (cf. [6]), $[D^\times, D^\times] = \widetilde{E}(D) =$ the kernel of the reduced norm. Hence the well known Eichler-Kneser Strong Approximation Theorem [2],[4] (and its analog due independently to Morita [8] and Swan [9], when $K$ is a function field with $\#K_0 < \infty$) implies

(SAT) (a) for $D$ over $R(P) \Leftrightarrow$ (EC) for $D$ over $R(P)$.

## 0.4

Apart from global fields, we shall prove;

**Theorem 2** (proof in 3.4) *For any PF-field $K$,*

(a$''$) *for $D$ over $R(P)$* $\Rightarrow$ (EC) *for $D$ over $R(P)$.*

All in all, the most optimistic speculation would be "(a) $\Leftrightarrow$ (a$'$) $\Leftrightarrow$ (a$''$) $\Leftrightarrow$ (EC)" for any central division algebras over any PF-fields. In this direction we can extend our previous result [11] as,

**Theorem 3** (proof in 4.4) *When $K$ is an algebraic function field of one variable over the reals,*

(EC) *for $D$ over $R(P)$* $\Rightarrow$ (a) *for $D$ over $R(P)$.*

## 1. Idele Topology

Let $R$ be a Dedekind domain with the quotient field $K$. A finitely generated $R$-module $L$ is called an *$R$-lattice*, if it is torsion free (or equivalently projective) over $R$, then $K \otimes_R L$ is a finite dimensional $K$-vector space and by the natural embedding $L \to K \otimes L$, one can identify as $K \otimes L = KL$. An $R$-algebra $\Lambda$ is called an *$R$-order* if it is an $R$-lattice, then $K\Lambda = K \otimes \Lambda$ is a finite dimensional $K$-algebra. When a finite dimensional $K$-algebra $B$ is given, we call that $\Gamma$ is an *$R$-order of $B$*, if $\Gamma$ is an $R$-order and $B = K\Gamma$.

For a maximal ideal $p$ of $R$, let $R_p$ *always denote the $p$-adic completion of $R$.* Let $\widehat{R} := \prod R_p$, the product over all maximal ideals of $R$. By the diagonal embedding $R \to \widehat{R}$, $\widehat{R}$ is an $R$-algebra which is faithfully flat as an $R$-module. For any $R$-module $M$, put

$$M_p := M \otimes_R R_p, \qquad \widehat{M} := M \otimes_R \widehat{R}.$$

We shall be concerned with only the following two special cases.

1) $\Gamma$ is an $R$-order: Then, since $\Gamma$ is finitely generated projective $R$-module, $\widehat{\Gamma} := \Gamma \otimes_R \prod R_p \simeq \prod(\Gamma \otimes_R R_p) = \prod \Gamma_p$.

2) $B$ is a finite dimensional $K$-algebra: Then $\widehat{B} := B \otimes_R \widehat{R} \simeq B \otimes_K K \otimes_R \widehat{R} \simeq B \otimes_K \widehat{K}$, and since $\widehat{R}$ is faithfully flat over $R$, one may canonically view as $\widehat{B} \supset \widehat{\Gamma}, B$ and $B \cap \widehat{\Gamma} = \Gamma$ Moreover, there is a natural identification $\widehat{B} \simeq \varinjlim \widehat{\Gamma}/r\, (r \in R \setminus \{0\}) \simeq \prod' B_p$ (w.r.t $\Gamma_p$), where the last term denote the restricted direct product i.e. $\prod' B_p$ (w.r.t. $\Gamma_p) := \{x = (x_p) \in \prod B_p | x_p \in \Gamma_p$ for almost all $p\}$. The *adele topology* on $\widehat{B}$ is defined as the unique topology which induces on $\widehat{\Gamma}$ the direct prduct of $p$-adic topology $\prod \Gamma_p$, for one (hence any) $R$-order $\Gamma$ of $B$. The name comes from the fact that $\widehat{K}$ with this topology is called the (restricted) adele ring of $K$.

The *idele topology* in $\widehat{B}^\times$ is defined as the unique topology which induces on $\widehat{\Gamma}^\times$ the direct product of $p$-adic topology $\prod \Gamma_p^\times$, for one (hence any) $R$-order $\Gamma$ of $B$. The following explicit description of the idele topology will be useful for us.

## 1.1

For any $R$-order $\Gamma$ of $B$ and non-zero $r \in R$, put

$$(0) \quad \begin{cases} U_p(\Gamma, r) := \Gamma_p^\times \cap (1 + r\Gamma_p) = \begin{cases} \Gamma_p^\times & \text{if} \quad r \in R_p^\times \\ 1 + r\Gamma_p & \text{if} \quad r \in pR_p. \end{cases} \\ U(\Gamma, r) := \prod_p U_p(\Gamma, r) = \widehat{\Gamma}^\times \cap (1 + r\widehat{\Gamma}), \\ \Gamma(r) := R + r\Gamma, \text{ which is an } R\text{-order of } B \text{ again.} \end{cases}$$

By definitions,we have

(1) $\{U(\Gamma, r) | r \in R \setminus \{0\}\}$ is a fundamental system of neighbourhoods of 1 in $\widehat{B}^\times$ in the idele topology (for any one fixed $\Gamma$).

(1′) $\{r\widehat{\Gamma} | r \in R \setminus \{0\}\}$ is a fundamental system of neighbourhoods of 0 in $\widehat{B}$ in the adele topology.

Let $H$ be a subgroup of $\widehat{B}^\times$ and $\overline{H}$ will denote the closure of $H$ in $\widehat{B}^\times$

(2) If $H \cap (1 + r\widehat{\Gamma}) \subset \widehat{\Gamma}^\times$ for some $\Gamma$ and $r \in R \setminus \{0\}$, (in particular if $H \cap \widehat{\Gamma} \subset \widehat{\Gamma}^\times$), then the idele topology of $\widehat{B}^\times$ and the adele topology of $\widehat{B}$ induce the same topology on $H$. Indeed, $H \cap U(\Gamma, rr') = H \cap (1 + rr'\widehat{\Gamma})$ for any $r' \in R \setminus \{0\}$.

Since $\Gamma(r)_p^\times = R_p^\times U_p(\Gamma, r)$, we have

(3) $\widehat{R}^\times U(\Gamma, r) = \widehat{\Gamma(r)}^\times$

(4) If $\widehat{R}^\times \subset \overline{H}$, then $HU(\Gamma, r) \supset \widehat{R}^\times$ so that $\overline{H} = \bigcap_{r\neq 0} H\widehat{\Gamma(r)}^\times = \bigcap_{r\neq 0} \widehat{\Gamma(r)}^\times H$.

## 1.2 Proof of Proposition 2 §0.

For any $R$-order $\Gamma$ of $B$, put $\Lambda = \Gamma^{op}$, the opposite ring of $\Gamma$ and $L := \Gamma$. then $\mathrm{End}_\Lambda L = \Gamma$ Hence the condition ( (c$''$) for any $L$ such that $K\mathrm{End}_\Lambda L \simeq B$) is equivalnet with the condition $\widetilde{E}(\widehat{B}) \subset \widehat{\Gamma}^\times B^\times$ for any $\Gamma$ But we have $\bigcap_\Gamma \widehat{\Gamma}^\times B^\times = \overline{\widehat{R}^\times B^\times}$. since $\widehat{\Gamma}^\times B^\times$ is closed and contains $\widehat{R}^\times B^\times$, so $\overline{\widehat{R}^\times B^\times} \subset \bigcap_\Gamma \widehat{\Gamma}^\times B^\times \subset \bigcap_{r\neq 0} \widehat{\Gamma(r)}^\times B^\times$ while we have $\overline{\widehat{R}^\times B^\times} = \bigcap_r \widehat{\Gamma(r)}^\times B^\times$, by (4).

## 1.3 Results of Vaserstein.

Let $A$ be a ring with 1, and $E_n(A)$ be the elementary subgroup of $GL_n(A) := M_n(A)^\times$ By the usual embedding $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1_{n-1} \end{pmatrix}$, we consider as $A^\times = GL_1(A) \subset GL_n(A)\,(n \geq 2)$. Let $\widetilde{E}(A)$ be the group of Vaserstein, i.e. the subgroup of $A^\times$ given by the generators as

$$\widetilde{E}(A) := \left\langle (1+xy)(1+yx)^{-1} \,|\, x, y \in A, 1+xy \in A^\times \right\rangle$$

The commutator subgroup $[A^\times, A^\times]$ is always contained in $\widetilde{E}(A)$. Further, if $A$ is local, $\widetilde{E}(A) = [A^\times, A^\times]$.

If $A$ is semi-local, the well known Lemma of Bass and the fundamental results of Vaserstein ([10], Th.3.6) state:

(5) $GL_n(A) = A^\times E_n(A)\,(n \geq 2)$.

(6) $A^\times \cap E_n(A) = \widetilde{E}(A)\,(n \geq 2)$.

## 1.4

**Lemma** *Let $B$ be a finite dimensional $K$-algebra and $\Gamma$ be an $R$-order of $B$. Then the equality (5) of Bass (resp. (6) of Vaserstein) holds for $A = \widehat{B}$ or $\widehat{\Gamma}$. (where $\widehat{B}$ or $\widehat{\Gamma}$ is* *not semi-local if $R$ is not semi-local.)*

**Proof** In the proof of [10] Th.3.6 (a), where semi-locality of $A$ is assumed, it is in fact proved that

(i) If the ring $A$ satisfies the following condition (5′), then (5) holds.

(5′) For any finitely generated left ideal $L$ and $x \in A$,

$$Ax + L = A \Rightarrow (x + L) \cap A^\times \neq \phi.$$

(ii) If $A$ satisfies (5′) and moreover the following (6′), then (6) holds.

(6′) $Ax_1 + Ax_2 = A \Rightarrow {}^\forall y \in A, {}^\exists v, q, u \in A$ such that $x_1 + vx_2 \in A^\times, 1 - yqv \in A^\times$, $x_1 + u(x_2 + yx_1) \in A^\times$, $x_1 + u(x_2 + yqx_1) \in A^\times$

Now, let $A = \prod' A_p$ (w.r.t $C_p$) be the restricted direct product of $A_p$ with respect to its subring $C_p$, over some index $p$'s. If each $A_p, C_p$ satisfies (5′) and (6′), it is easy to see that $A$ itself satisfies (5′) and (6′). This applies for $\widehat{B}$ or $\widehat{\Gamma}$, since $B_p$ and $\Gamma_p$ are semi-local.

## 1.5 Proof of Proposition 1 §0.

As is well known (cf [3] §2 and §3), the property (c′) is equivalent with the following

(c‴) $\widehat{B}^\times \cap GL_n(B)GL_n(\widehat{\Gamma}) = B^\times \widehat{\Gamma}^\times$ for any $n \geq 2$.

By 1.4, we have

1) $GL_n(B) = B^\times E_n(B)$ 2) $GL_n(\widehat{\Gamma}) = E_n(\widehat{\Gamma})\widehat{\Gamma}^\times$

Since $E_n(B)$ is dense in $E_n(\widehat{B})$ in the idele topology of $\widehat{B}^\times$ (cf [3] 1.2.1)

3) $E_n(B)GL_n(\widehat{\Gamma}) = E_n(\widehat{B})GL_n(\widehat{\Gamma})$.

Using 1), 3), 2) in this order, we have: $GL_n(B)GL_n(\widehat{\Gamma}) = B^\times E_n(B)GL_n(\widehat{\Gamma}) = B^\times E_n(\widehat{B})GL_n(\widehat{\Gamma}) = B^\times E_n(\widehat{B})E_n(\widehat{\Gamma})\widehat{\Gamma}^\times = B^\times E_n(\widehat{B})\widehat{\Gamma}^\times$

Hence, the left hand side of (c‴)$= \widehat{B}^\times \cap B^\times E_n(\widehat{B})\widehat{\Gamma}^\times = B^\times(\widehat{B}^\times \cap E_n(\widehat{B}))\widehat{\Gamma}^\times = B^\times \widetilde{E}(\widehat{B})\widehat{\Gamma}^\times$, the last equality by 1.4 again. This implies that (c‴) is equivalent with (c″).

## 1.6 Change of the base field.

Let $K'$ be a finite extension field of $K$ contained in the center of $B$. and let $R'$ be the integral closure of $R$ in $K'$ Then $R'$ is a Dedekind domain with the quotient $K'$ and $B$

is a finite dimensional $K'$-algebra. Assume the following condition

(f) $R'$ is a finitely generated $R$-module.

Then there are canonical isomorphism $\widehat{R'} \simeq R' \otimes_R \widehat{R}$ and $K' \otimes_{R'} \widehat{R'} \simeq K' \otimes_R \widehat{R}$ (cf. [7] Th.1 and Prop.4 Chap. II §3), so that $B \otimes_{R'} \widehat{R'} \simeq B \otimes_R \widehat{R}$ including the topology. Hence the approximation property (a) (resp. (a$'$)) of $B$ over $R$ is equivalent with that of $B$ over $R'$ and (a$''$) over $R$ implies that over $R'$

(i) For a residually separable algebra $B$ (i.e. $B/J(B)$ is separable) the $B^\times$- approximation problem is reduced, by Theorem 1, to that of a central division algebra.

(ii) If $K$ is a PF-field, the condition (f) always holds (cf. [5] Th.72), so that we get the reduction to a central division algebra even for residually inseparable case.

## 2. Reduction to a Division Algebra.

Let $B$ be a finite dimensional $K$-algebra with the Jacobson radical $J = J(B)$, $\varphi : B \to B' := B/J$ be the canonical $K$-morphism and $\Gamma' := \varphi(\Gamma)$. Then $\Gamma'$ is an $R$-order in $B'$, and $\varphi$ induces the following surjective morphisms: $\varphi_0 : \Gamma \to \Gamma'$, $\widehat{\varphi} := \varphi \otimes 1 : \widehat{B} = B \otimes \widehat{R} \to B' \otimes \widehat{R} = \widehat{B}'$ and $\widehat{\varphi}_0 := \varphi_0 \otimes 1 : \widehat{\Gamma} = \Gamma \otimes \widehat{R} \to \Gamma' \otimes \widehat{R} = \widehat{\Gamma}'$

Since $\widehat{R}$ is faithfully flat over $R$,

1) $\mathrm{Ker}\varphi_0 = \Gamma \cap J \subset J(\Gamma)$, $\mathrm{Ker}\widehat{\varphi} = J \otimes \widehat{R} = \widehat{J} \subset J(\widehat{B})$, $\mathrm{Ker}\widehat{\varphi}_0 = \widehat{\Gamma \cap J} \subset J(\widehat{\Gamma})$.

2) Viewing as $\widehat{B} \supset \widehat{\Gamma}, B$ and $\widehat{\Gamma} \cap B = \Gamma$, $\varphi_0, \widehat{\varphi}_0, \varphi$ is the restriction of $\widehat{\varphi}$ to $\Gamma, \widehat{\Gamma}, B$ respectively.

By 1), $1 + \widehat{J} \subset \widehat{B}^\times$ so that $\widehat{\varphi}$ induces the exact sequence of groups:

3) $1 \to 1 + \widehat{J} \to \widehat{B}^\times \to \widehat{B}'^\times \to 1$, and $\widehat{\varphi}^{-1}(\widehat{B}'^\times) = \widehat{B}^\times$

Consequently, we have

4) $\widehat{\varphi}(\widetilde{E}(\widehat{B})) = \widetilde{E}(\widehat{B}')$.

By the same reasoning, we have

5) $\widehat{\varphi}(\widetilde{E}(B)) = \widetilde{E}(B')$.

Also we have

6) $\widehat{\Gamma}^\times = \widehat{\varphi}_0^{-1}(\widehat{\Gamma}'^\times)$, which in turn implies

7) $\widehat{\varphi}(U(\Gamma, r)) = U(\Gamma', r)$, in the notation of 1.1.

## 2.1

**Lemma** *Let $H$ be a subgroup of $\widehat{B}^\times$ and $\overline{H}$ be its closure in $\widehat{B}^\times$*

*(i)* $\widetilde{E}(\widehat{B}) \subset \overline{H} \Rightarrow \widetilde{E}(\widehat{B}') \subset \overline{\widehat{\varphi}(H)}$

*(ii) If $1 + \widehat{J} \subset \overline{H}$, then the converse implication ($\Leftarrow$) also holds.*

*(iii)* $1 + \widehat{J} \subset \overline{B^\times}$

**Proof** (i) and (ii): $(\widetilde{E}(\widehat{B}) \subset \overline{H}) \overset{(1)\ 1.1}{\Longleftrightarrow} (\widetilde{E}(\widehat{B}) \subset HU(\Gamma, r)$ for any $r \in R \setminus \{0\})$ $\overset{4)\ \&\ 7)}{\Longrightarrow} (\widetilde{E}(\widehat{B}') \subset \widehat{\varphi}(H)U(\Gamma', r))$ for any $r \in R \setminus \{0\})$ $\overset{3),\ 4)\ \&\ 7)}{\Longrightarrow} (\widetilde{E}(\widehat{B}) \subset (1 + \widehat{J})HU(\Gamma, r)$ $(= HU(\Gamma, r)$ if $\overline{H} \supset 1 + \widehat{J})$ for any $r \in R \setminus \{0\})$.

(iii) Since any element of $\widehat{J}$ is nilpotent, $(1 + \widehat{J}) \cap (1 + r\widehat{\Gamma}) = 1 + (\widehat{J} \cap r\widehat{\Gamma}) \subset \widehat{\Gamma}^\times$, hence by (2) 1.1, the idele topology on $1 + \widehat{J}$ is induced from the adele topology. Since $J$ is dense in $\widehat{J}$ in the adele topology, $1 + J$ is dense in $1 + \widehat{J}$ in the idele topology so that $1 + \widehat{J} \subset (1 + J)U(\Gamma, r) \subset B^\times U(\Gamma, r)$ for any $r \in R \setminus \{0\}$.

## 2.2

**Lemma** *Let $B = \overset{m}{\underset{i=1}{\oplus}} B_i$ be the ring direct sum of finite dimensional $K$-algebras. Then we have the following implications.*

*(i)* (a) *(resp.* (a')*) for $B \Leftrightarrow$* (a) *(resp.* (a')*) for any $B_i\,(1 \le i \le m)$.*

*(ii)* (a'') *for $B \Rightarrow$* (a'') *for any $B_i\,(1 \le i \le m)$.*

**Proof** Let $\Gamma_i$ be an $R$-order of $B_i$, then $\Gamma := \oplus\Gamma_i$ is an $R$-order of $B$. By the canonical isomorphism $\widehat{B} = B \odot \widehat{R} \simeq \oplus(B_i \odot \widehat{R}) = \oplus\widehat{B}_i$, $\widehat{B}^\times \simeq \prod \widehat{B}_i^\times$, $\widehat{\Gamma}^\times \simeq \prod \widehat{\Gamma}_i^\times$, $U(\Gamma, r) \simeq \prod U(\Gamma_i, r)$, $\widetilde{E}(B) \simeq \prod \widetilde{E}(B_i)$ and $\widetilde{E}(\widehat{B}) \simeq \prod \widetilde{E}(\widehat{B}_i)$, the claims are completely obvious.

## 2.3 Proof of Theorem 1 §0.

Put $B_i = M_{n_i}(D_i)$, $n_i = 1\,(1 \le i \le r)$, $n_i \ge 2\,(r < i \le m)$. Recall that (a) holds for

$B_i\,(r < i \leq m)$ ((3) of 0.1) and apply 2.1 and 2.2, then we get the following implications which obviously prove Theorem 1.

(a) for $B \Rightarrow$ (a) for $B' \Leftrightarrow$ (a) for $D_i\,(1 \leq i \leq r)$

(a$'$) for $B \Leftrightarrow$ (a$'$) for $B' \Leftrightarrow$ (a$'$) for $D_i\,(1 \leq i \leq r)$

(a$''$) for $B \Leftrightarrow$ (a$''$) for $B' \Rightarrow$ (a$''$) for $D_i\,(1 \leq i \leq r)$.

## 3. (a$''$) $\Rightarrow$ (EC) for a PF-field.

Let $K$ be a PF-field in the sense of [1], $D$ be a central division $K$-algebra of dimension $n^2$, $[D : K] = n^2$ Let $D_v := D \otimes_v K_v$ be the completion at $v \in \mathfrak{V}$. Let $\mathfrak{N} : D \to K$ be the reduced norm and $\mathfrak{N}_v : D_v \to K_v$ be its extension.

If $D_v$ is a division algebra, $D_v \ni x \mapsto |\mathfrak{N}_v x|_v^{1/n}$ defines a norm of $D_v$ as a $K_v$-vector space. While for any basis $\{e_i | 1 \leq i \leq n^2\}$ of $D$ over $K$, writing $x = \sum \xi_i e_i \in D_v$, $x \mapsto \operatorname*{Max}_i |\xi_i|_v$ is also a norm of $D_v$. Hence there is a constant $c_v > 0$ such that

(1) $\quad \operatorname*{Max}_i |\xi_i|_v \leq c_v |\mathfrak{N}_v x|_v^{1/n} \quad (x = \sum \xi_i e_i)$.

For almost all $v$, we have: $v$ is non-archimedean; $\{\sum \xi_i e_i || \xi_i|_v \leq 1\}$ is a maximal order of $D_v$; $|\det Tr(e_i e_j)|_v = 1$. Hence for almost all $v$ such that $D_v$ is a division algebra, $D_v/K_v$ is unramified and $|\mathfrak{N}_v x|^{1/n} = \operatorname*{Max}_i |\xi_i|_v$. Thus we can choose $c_v$ as

(1$'$) $\quad c_v = 1$ for almost all $v$ such that $D_v$ is a division algebra.

Let $R$ be a Dedekind domain with the quotient field $K$, so that it has the form $R = R(P) := \{\xi \in K || \xi|_p \leq 1$ for any $p \in P\}$ by some non-empty proper subset $P$ consisting of non-archimedean valuation of $\mathfrak{V}$. For a fixed $R$, we can obviously choose a basis $\{e_i | 1 \leq i \leq n^2\}$ satisfying

(2) $\quad \Gamma := \sum_{i=1}^{n^2} R e_i$ is an $R$-order of $D$. and $e_1 = 1$.

Then $\Gamma(r) := R + r\Gamma$ is also an $R$-order for any $r (\neq 0) \in R$.

### 3.1

**Lemma** *Assume that $D$ does not satisfy the Eichler's condition* (EC) *over* $R = R(P)$, *i.e. the following* $\neg$(EC) *is satisfied.*

$\neg$(EC): *$D_v$ is a division algebra for any $v \in \mathfrak{V} \setminus P$*

*(i) Let $\{e_i\}$ be a basis of $D$ satisfying (2), then there is a positive constant $c$ depending only on $\{e_i\}$ but not on $r(\neq 0) \in R$ such that*

$$\prod_P |r|_p < c \Rightarrow \Gamma(r)^\times = R^\times$$

*(ii) $\widehat{R}^\times D^\times$ is closed in $\widehat{D}^\times$*

**Proof** (i) It suffices to take $c := \prod_{\mathfrak{V}\setminus P} c_v^{-1}$ (which is well defined by $(1')$). Indeed, if $\Gamma(r)^\times \neq R^\times$, there is some $x = \sum \xi_i e_i \in \Gamma(r)^\times$ with $\xi := \xi_i \neq 0$ for some $i \geq 2$. At $p \in P$, since $x \in \Gamma(r)^\times$ so that $|\mathfrak{N}_p x|_p = 1$, we have

(3) $|\xi|_p \leq |r|_p = |r|_p |\mathfrak{N}x|_p^{1/n}$

Using the product formula, (1) at $v \in \mathfrak{V} \setminus P$ and (3) at $p \in P$, the product formula again, in this order, we get

$$1 = \prod_{\mathfrak{V}} |\xi|_v = \prod_{\mathfrak{V}\setminus P} |\xi|_v \times \prod_P |\xi|_p \leq \prod_{\mathfrak{V}\setminus P} c_v |\mathfrak{N}x|_v^{1/n} \times \prod_P |r|_p |\mathfrak{N}x|_p^{1/n}$$

$$= \prod_{\mathfrak{V}\setminus P} c_v \times \prod_P |r|_p = c^{-1} \prod_P |r|_p.$$

(ii) Put $R(c) := \{r \in R\setminus\{0\} | \prod_P |r|_p < c\}$. If $r \in R(c)$, by (i), we have $\widehat{\Gamma}(r)^\times \cap D^\times = \Gamma(r)^\times = R^\times$ This obviously implies

(4) $\bigcap_{r \in R(c)} (D^\times \widehat{\Gamma(r)}^\times) = D^\times (\bigcap_{r\in R(c)} \widehat{\Gamma(r)}^\times)$.

Then together with (4) 1.1, we have

$$\overline{D^\times \widehat{R}^\times} = \bigcap_{r\neq 0} (D^\times \widehat{\Gamma(r)}^\times) \subset \bigcap_{r\in R(c)} (D^\times \widehat{\Gamma(r)}^\times) = D^\times (\bigcap_{r\in R(c)} \widehat{\Gamma(r)}^\times) = D^\times \widehat{R}^\times \subset \overline{D^\times \widehat{R}^\times}$$

## 3.2

As usual, we consider $D_p^\times$ as the subgroup of $\widehat{D}^\times$ consisting of the elements $x = (x_p) \in \widehat{D}^\times$ such that $x_q = 1$ for $q \in P \setminus \{p\}$, Under this convention, the following is obvious.

(5) $\quad \#P \geq 2 \Rightarrow \widehat{R}^\times D^\times \cap D_p^\times \subset K_p^\times$

If $\#P < \infty$, then $R$ is semi-local and $\overline{D^\times} = \widehat{D}^\times$, hence 3.1 implies

(6) $\quad 2 \leq \#P < \infty \Rightarrow$ (EC).

Indeed: $\neg$(EC) implies $\overline{\widehat{R}^\times D^\times} = \widehat{R}^\times D^\times$ so that $\widehat{D}^\times \subset \widehat{R}^\times D^\times$ hence $D_p^\times \subset D_p^\times \cap \widehat{R}^\times D^\times \subset K_p^\times$ a contradiction to the assumption that $D$ is non-commutative.

### 3.3

<u>**Lemma**</u> *Let $D$ be a central division algebra over a PF-field $K$. Then $D_v$ is not a division algebra for infinitely many $v \in \mathfrak{V}$.*

<u>**Proof**</u> If $\mathfrak{V}$ contains at least one archimedean valuation (i.e. if $K$ is a number field), as is well known, much stronger results are known. Assume that $\mathfrak{V}$ consists of non-archimedean valuations. If $\#\{v \in \mathfrak{V} | D_v$ is not a division algebra$\} < \infty$, then obviously we can choose a subset $P$ of $\mathfrak{V}$ such that $2 \leq \#P < \infty$ and $\neg$(EC), a contradiction with (6) 3.2.

### 3.4 Proof of Theorem 2

We shall prove:

$\neg(\mathrm{EC}) \Rightarrow [\widehat{D}^\times, \widehat{D}^\times] \not\subset \overline{\widehat{R}^\times D^\times}$

Suppose not, then $[\widehat{D}^\times, \widehat{D}^\times] \subset \widehat{R}^\times D^\times$ by 3.1, so that $[D_p^\times, D_p^\times] = D_p^\times \cap [\widehat{D}^\times, \widehat{D}^\times] \subset D_p^\times \cap \widehat{R}^\times D^\times \subset K_p^\times$ for any $p \in P$ It is a contradiction, since if $x, y$ do not commute in $D_p^\times$, then one of $[x, y]$ and $[x, 1+y]$ does not belong to $K_p^\times$

## 4. (EC) $\Rightarrow$ (a) for a Real Coefficient Case.

We shall derive our Theorem 3 from our previous result [11], where it is proved only for a special case of $K = \mathbb{R}(X)$. For this purpose, we prepare a few lemmas, which are of quite general nature, but regretfully, effectively applicable only for a very restricted situation like in Theorem 3, so that we state them only for PF-fields.

## 4.1

Let $D$ be a central division algebra over a PF-field $K$ and $R = R(P)$ as in 0.3. For a fixed $p_0 \in P$, as usual, we identify $D_{p_0}^\times$ as the (closed normal) subgroup of $\widehat{D}^\times$, consisting of elements $x = (x_p) \in \widehat{D}^\times \subset \prod D_p^\times$ with $x_p = 1$ for $p \neq p_0$. Then $\{\widetilde{E}(D_p)|p \in P\}$ generates a dense subgroup of $\widetilde{E}(\widehat{D})$ in $\widehat{D}^\times$ (cf. [2] §51). Hence a closed subgroup $H$ of $\widehat{D}^\times$ contains $\widetilde{E}(\widehat{D})$ if and only if it contains $\widetilde{E}(D_p) = [D_p^\times, D_p^\times]$ for all $p \in P$ By the Chinese Remainder Theorem, 'all' can be replaced by 'almost all' In particular we have:

(1) (a) for $D$ over $R \Leftrightarrow [D_p^\times, D_p^\times] \subset \overline{\widetilde{E}(D)}$ for almost all $p$,

and the corresponding (1′) (resp. (1″)) for (a′) (resp. (a″)).

Let $K'$ be a finite extension field of $K$, and let $P'$ be the set of all (non-equivalent) valuations of $K'$ lying over$P$, $P' = \{p'|p' \supset p,\, p \in P\}$. The integral closure $R'$ of $R$ in $K'$ is given by $R' = \{0\} \cup \{x \in K'^\times || x|_{p'} \leq 1 \text{ for any } p' \in P'\}$.

Put $D' := D \otimes_K K'$ By 1.6, $\widehat{D'} := D' \otimes_{R'} \widehat{R'} \simeq D' \otimes_R \widehat{R} \supset D \otimes_R \widehat{R} = \widehat{D}$ as topological rings, and

(2) $\widehat{D'}^\times \supset \widehat{D}^\times,\ \widehat{D'}^\times \supset \prod_{p' \supset p} D'^\times_{p'} \simeq D'^\times_p \supset D_p^\times$ as topological groups.

In the following $\overline{(\ )}$ denotes the closure in $\widehat{D'}^\times$

Let consider the following condition $(*)$.

$(*)$ For almost all $p \in P$ $p' \supset p \Rightarrow [D'^\times_{p'}, [D_p^\times, D_p^\times]] = [D'^\times_{p'}, D'^\times_{p'}]$.

<u>**Lemma**</u> *Assume that the condition $(*)$ holds. Then*

(a″) *for $D$ over $R \Rightarrow$* (a) *for $D'$ over $R'$*

<u>**Proof**</u> By the Chinese Remainder Theorem, $D'^\times$ is dense in $\prod_{p' \supset p} D'^\times_{p'}$ Hence, by (2), $[D'^\times_{p'}, [D_p^\times, D_p^\times]] \subset \overline{[D'^\times, [D_p^\times, D_p^\times]]}$, so that the assumption $(*)$ implies

(3) $[D'^\times_{p'}, D'^\times_{p'}] \subset \overline{[D'^\times\ [D_p^\times, D_p^\times]]}$ for almost all $p \in P$

On the other hand we have

(a$''$) for $D$ over $R \overset{(1'')}{\Longleftrightarrow} [D_p^\times, D_p^\times] \subset \overline{\widehat{R}^\times D^\times}$ for almost all $p \in P \Rightarrow [D'^\times, [D_p^\times, D_p^\times]] \subset [D'^\times, \overline{\widehat{R}^\times D^\times}] \subset \overline{[D'^\times, D^\times]} \subset \overline{[D'^\times, D'^\times]} = \widetilde{E}(D')$.

Hence by (3), we have $[D_{p'}'^\times, D_{p'}'^\times] \subset \overline{\widetilde{E}(D')}$ for almost all $p$, which is equivalent with ((a) for $D'$ over $R'$) by (1).

## 4.2

Now assume that the constant field $K_0 = \mathbb{R}$, i.e. $K$ is an algebraic function field of one variable over the reals.

Recall from [11] that $Br(K) \simeq K^\times/\mathfrak{N}(K(\sqrt{-1})^\times) = K^\times/(K^2 + K^2) \cap K^\times$, so that any central division algebra $D$ over $K$ is a quaternion algebra of the form $D \simeq \{-1, f\}$ with $f \in K^\times$ $D$ is trivial if and only if $f \in K^2 + K^2$

We call a valuation $v \in \mathfrak{V}$ real (resp. imaginary) if the residue field is isomorphic to $\mathbb{R}$ (resp. $\mathbb{C}$). $K(\sqrt{-1})$ is an algebraic function field of one variable over $\mathbb{C}$, so the corresponding $\mathfrak{V}'$ is identified with the Riemann surface $\mathfrak{R}$, and $K(\sqrt{-1})$ with the field of all meromorphic functions on $\mathfrak{R}$. Since a real valuation $v$ of $K$ does not decompose on $K(\sqrt{-1})$, the set $RP(K)$ of all real valuations can be embedded in $\mathfrak{R}$ as a finite disjoint union of closed curves. Then we have

$$K = \{\varphi \in K(\sqrt{-1}) | \varphi(z) \in \mathbb{R} \text{ for } z \in RP(K)\}.$$

Furthermore, as shown in [11],

$$K^2 + K^2 = \{f \in K | f(z) \geq 0 \text{ for } z \in RP(K)\},$$

so $\{-1, f\}$ is trivial for such $f$

Let $P$ be a non-empty proper subset of $\mathfrak{V}$.

**Lemma** *If $D$ satisfies* (EC) *over $R(P)$. then $D$ can be written as $D = D_0 \otimes_{\mathbb{R}(g)} K$, where $g \in R(P) \setminus \mathbb{R}$ and $D_0$ is a central division $\mathbb{R}(g)$-algebra satisfying* (a) *over $\mathbb{R}[g]$.*

**Proof** (EC) for $D$ means that $D_{v_0}$ is trivial for some $v_0 \in \mathfrak{V} \setminus P$ From Riemann-Roch Theorem, for any $f \in K^\times$ we can find $h \in K^\times$ such that $g := h^2 f$ has the unique pole at $v_0$. Therefore $D$ can be written as $D = \{-1, g\}$, where $g \in R(P)$ and has the unique pole at $v_0$.

Since $D_{v_0}$ is trivial, we have either (i) $v_0$ is imaginary or (ii) $v_0$ is real and $g$ is positive around $v_0$. In any case, $g$ is bounded from below on $RP(K)$, since $g$ has no pole other than $v_0$. So, $g + c \in K^2 + K^2$ for some $c \in \mathbb{R}$, hence $D = \{-1, g\} = \{-1, g(g+c)\} \simeq D_0 \otimes_{\mathbb{R}(g)} K$ where $D_0 = \{-1, g(g+c)\}$ over $\mathbb{R}(g)$ which satisfies (EC) over $\mathbb{R}[g]$ since $X(X+c)$ is monic and quadratic. From our previous result [11], $D_0$ satisfies (a) over $\mathbb{R}[g]$.

## 4.3

**Lemma** *If $K$ is an algebraic function field of one variable over $\mathbb{R}$, then the condition (*) in 4.1 is satisfied for any $D$.*

**Proof** Note that $D_p$ is unramified for almost all $p \in P$ If $D_p$ is trivial, then $D_p^\times = GL(2, K_p)$ and $[D_p^\times, D_p^\times] = SL(2, K_p)$. In this case $[D_{p'}'^\times, [D_p^\times, D_p^\times]] = [GL(2, K_{p'}'), SL(2, K_p)]$ is a normal subgroup of $SL(2, K_{p'}')$ not contained in its center, so it must coincide with $SL(2, K_{p'}')$.

If $D_p$ is an unramified quaternion algebra, then $p$ is real so that $-1 \notin K_p^2$ and $K_p^2 + K_p^2 = K_p^2$. Thus the reduced norm $\mathfrak{N}_p$: $D_p^\times \to K_p^\times$ maps $D_p^\times$ onto $K_p^{\times 2}$ with the kernel $[D_p^\times, D_p^\times]$. This implies $D_p^\times = K_p^\times[D_p^\times, D_p^\times]$, so that $[D_{p'}'^\times, [D_p^\times, D_p^\times]] = [D_{p'}'^\times, D_p^\times] \supset [D_p^\times, D_p^\times]$, hence the left hand side is a normal subgroup of $[D_{p'}'^\times, D_{p'}'^\times]$ containing $i \in [D_p^\times, D_p^\times]$, and as such it coincides with $[D_{p'}'^\times, D_{p'}'^\times]$. (Proof for $D_{p'}' \simeq \{-1, -1\}$ is as follows: Let $N$ be a normal subgroup of $[D_{p'}'^\times, D_{p'}'^\times]$ containing $i$, then $\{x \in D_{p'}' | x^2 + 1 = 0\} \subset N$ since such $x$ is conjugate with $i$ by Skolem-Noether Theorem. So for any $a \in K_{p'}'$ such that $1 - a^2 \in K_{p'}'^2$, we have $-ai + bj \in N$ (with $a^2 + b^2 = 1$), hence $y := i(-ai + bj) = a + bij \in N$ which satisfies $y^2 - 2ay + 1 = 0$. Thus, again from Skolem-Noether Theorem, every $y \in [D_{p'}'^\times, D_{p'}'^\times]$ belongs to $N$).

## 4.4 Proof of Theorem 3 §0

Assume that $D$ satisfies (EC) over $R(P)$. Applying Lemmas 4.1 and 4.3 to the result of Lemma 4.2 (regarding $\mathbb{R}(g)$ as $K$ and $K$ as $K'$), we see that $D$ satisfies (a) over $\mathbb{R}[g]_K$, the integral closure of $\mathbb{R}[g]$ in $K$. Since $g \in R(P)$, we have $R(P) \supset \mathbb{R}[g]_K$ so that (a) over $\mathbb{R}[g]_K$ implies (a) over $R(P)$.

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